Validity of basic concepts in nonlinear cooperative Fokker-Planck models

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The stochastic amplification of a periodic signal in a truly nonlinear Fokker-Planck model, whose drift coefficient exhibits a functional dependence on the distribution function, is analyzed numerically by means of a finite-difference method. Our aim is to check the validity of basic concepts widely used in studying linear and/or undriven systems. A perturbation approach to numerically evaluate the generalized susceptibility of the model by means of the linear response theory is tested and found to be adequate for weak driving fields. We also check the validity of the Floquet theory and the H theorem for which no extension to the case of truly nonlinear driven systems exists. The influence of the functional nonlinearity on the typical stochastic resonance effects is pointed out. [S1063-651X(96)13009-9]

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I. INTRODUCTION

The response of dynamical systems is an active field of study, both at the theoretical and experimental levels. In particular, a large amount of work has been recently devoted to understanding the phenomenon of stochastic resonance [1,2]. In its simplest form, stochastic resonance occurs in a bistable system driven by a time periodic external field. A typical model repeatedly studied by many authors within this context is that governed by the Langevin equation (see, e.g., [1–6])

$$\dot{x}(t)=x-x^3-A\cos\Omega t+\eta(t), \tag{1}$$

where $A\cos\Omega t$ represents the effect of the external signal and $\eta(t)$ is a Gaussian noise with zero mean and $\langle \eta(t)\eta(s)\rangle = 2D\delta(t-s)$. The corresponding Fokker-Planck equation for the probability density is

$$\partial_t P(x,t)=\partial_x(x^3-x+A\cos\Omega t+D\partial_x)P(x,t). \tag{2}$$

The dynamics is that of a driven Brownian particle moving in the symmetric bistable potential $U(x)=x^4/4-x^2/2$ in the large damping limit. The periodic field raises the potential wells alternatively, and this effect, together with the action of the stochastic term, makes the particle jump over the potential barrier. The resulting particle motion is coherent with the driving field. The phenomenon of stochastic resonance exists in the limit of weak driving amplitudes and for driving frequencies smaller than the intrawell relaxation frequency, so that the response of the system, measured by its long-time noise average, shows oscillations with amplitudes which can be much larger than the external amplitude. It exposes a qualitative aspect of the noise, which is usually blurred by its diffusive effect. Namely, noise can be looked upon as something useful, in the sense that it allows a weak input signal to be amplified. Stochastic resonance has recently been observed experimentally in a system where the noise was purely thermal [7], as well as in some biological systems [8–10].

The theoretical treatment of the problem can be carried out by making use of two important theorems: the H theorem, which ensures the existence of a uniquely determined long-time limit solution, which is time dependent, $P_\infty(x,t)$, and the Floquet theorem, which guarantees that $P_\infty(x,t)$ is periodic in time with the same period $T=2\pi/\Omega$ as the external force

$$P_\infty(x,t)=P_\infty(x,t+T). \tag{3}$$

(In the following, by quasiequilibrium solutions of the Fokker-Planck equation we will mean its long-time limit solutions, which are explicitly dependent on t.) In particular, an analysis of Eq. (2), based on Floquet eigenmode expansions, shows that, even though the autocorrelation function lacks the strong mixing property, the long-time limit time-periodic solution is always reached regardless of the initial condition [3,5].

Up to now, however, most of the theoretical approaches have been limited to linear Fokker-Planck models, although the linear case is not the generic case. In this paper, by nonlinear we mean Fokker-Planck equations with an explicit dependence of the drift coefficient on the distribution function, $P(x,t)$. This contrasts with the more conventional usage in the literature on stochastic resonance of the term ''nonlinear system,'' which refers to a nonlinear dependence of the drift on the state variable x. In order to avoid confusion, we will call ''truly nonlinear systems'' to those systems with a nonlinear dependence on the distribution function. It is often the case that models linear in the distribution function, are a very poor and crude approximation which fails to exhibit a lot of interesting phenomena inherent to real physical processes. As an example of such phenomena, one can mention phase transitions that are difficult to treat within the realm of a linear Fokker-Planck operator, while they arise very naturally in the truly nonlinear one [11].

It is often the case that the driving forces depend on the state of the system itself, thus leading to a truly nonlinear Fokker-Planck equation whose drift and diffusion coeffi-

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cients are functionals of the probability distribution. These equations play an important role in many branches of physical science, such as plasma physics [12], kinetic theory of gases [13], nucleation [14], or the theory of phase transitions [11]. Clearly, truly nonlinear models are much richer as they possess remarkable properties that are absent in linear ones. In particular, they may have many distinct equilibrium solutions. This feature of truly nonlinear systems is of particular interest within the context of stochastic resonance, but their study remains a sufficiently difficult and even impossible task in most cases. The reason is that, in general, equations nonlinear in the distribution function are not covered by many useful theorems often applied to linear equations, while their dynamics is usually beyond the powers of conventional, widely accepted, methods. For instance, it is the lack of a satisfactory analytical description for finite amplitudes and/or frequencies driving fields, which prevents ones from making use of the standard eigenmode expansions [3,5] and perturbation techniques [1,15], based on the Floquet theory and the H theorem. It is possible, of course, to treat nonlinear effects using perturbation expansions in terms of linear response theory, but any of these techniques involves approximations which limit their applicability to certain favorable regimes of parameter space [11,16]. On the other hand, numerically exact calculations for these problems are difficult. Simulations over very long time lengths are necessary, while the efficacy of conventional stochastic simulation methods [17] or iterative path integral schemes [18] is strongly limited by their hardly controllable accuracy. Moreover, one must be cautious on the use of stochastic simulation methods in the vicinity of phase-transition points that are precisely the ones at which the maximum of the amplitude due to stochastic resonance usually occurs. The reason is that the convergence of stochastic simulations, which is not so rapid by itself, becomes even slower in this domain due to critical slowing down. As a result, one has to generate a huge number of trajectories to reach an adequate level of accuracy. Otherwise ''unexpected'' phenomena may arise [19] (this problem will be discussed in more detail in Sec. III). Finally, we would like to emphasize that none of the above-mentioned techniques preserves automatically true quasiequilibrium solutions of the analytic equation, when they are numerically implemented.

It is, therefore, necessary to develop a simple computational tool for an accurate and error controlled treatment of the nonlinear dynamics that is formally rigorous but also amenable of extension to higher orders of approximation. In a preceding paper [20], such a technique has been developed by rearranging the Fokker-Planck operator so that it contains solely the second derivatives in x, and approximating these derivatives with a central difference by using a K-point Stirling interpolation formula. The finite-difference method developed has a fifth-order convergence in time and a $2K$th-order convergence in space, allowing us to reach an adequate level of accuracy with a mild increase in the number of grid points. The most appealing features of the method are that it is norm-conserved, and equilibrium-preserving in the sense that every equilibrium solution of the analytic equations is also an equilibrium solution of the discretized equations. The latter advantage is especially useful in studying stochastic resonance when just the knowledge of long-time solutions is necessary.

In the present work, we intend to study with this method the combined effects of noise and time periodic external forces in a truly nonlinear system which exhibits a phase transition. The nonlinearity is brought about by the fact that the overall system consists of very many subunits with mean-field interaction among them [21]. The model will be presented in Sec. II. In Secs. III and IV we check numerically the validity of the Floquet theory and the H theorem, respectively, that have not been proven yet within the context of nonlinear stochastic systems with periodic forcing. One may expect that a strong enough external field would be able to restore the ergodicity of the process, while the nonlinearity involved in the system would become an obstacle for the application of the Floquet theorem which is so often used in studying ordinary stochastic resonance in linear problems. Shiino studied the dynamical response of the nonlinear system of interest to a periodically oscillating external force [11]. In particular, using linear response theory, he was able to derive a simple expression for the generalized susceptibility $\chi(\omega)$,

$$\chi(\omega)=\frac{R(\omega)}{1-\theta R(\omega)}, \tag{4}$$

valid in the limit of weak driving forces, where θ represents the strength of the mean-field coupling among the subunits, and $R(\omega)$ is the generalized susceptibility of a linearized stochastic process. Only very recently this formula has been used by one of us [16] in order to numerically evaluate the response of the system to a very weak driving field without solving the truly nonlinear Fokker-Planck equation. Its validity, however, as well as the validity of calculations performed in [16] have not been tested yet by comparing with other numerically exact techniques whose utility is not restricted to the limit of weak driving fields solely. In Sec. V this comparison is done in terms of the present finite-difference scheme. Sec. VI contains some final remarks.

II. TRULY NONLINEAR DRIVEN SYSTEM

In this work, we consider a truly nonlinear driven system governed by

$$\partial_t P(x,t)=\partial_x[U'(x,t)+D\partial_x]P(x,t),$$
$$U(x,t)=\frac{x^4}{4}-\frac{x^2}{2}+\frac{\theta}{2}[x-\langle x(t)\rangle]^2+Ax\cos\Omega t, \tag{5}$$

where the prime denotes differentiation with respect to x, while the potential $U(x,t)$ depends upon the state of the system through the average $\langle x(t)\rangle$

$$\langle x(t)\rangle=\int_{-\infty}^{\infty}dx P(x,t)x. \tag{6}$$

It is seen that Eq. (5) reduces to the ordinary Fokker-Planck equation (2) with $\theta=0$. In the absence of the driving field ($A=0$) the model has been repeatedly studied by many authors as a typical case for an order-disorder phase transition.

The model deals with a system of infinitely many nonlinear coupled oscillators in the presence of an external white noise, and it was originally introduced by Kometani and Shimizu [22] to study self-organization processes in biological systems such as muscle contraction. A more complete statistical-mechanical treatment given later by Desai and Zwanzig [21] and by Dawson [23] pointed out its relation with the Weiss-Ising model. Since the dynamical issues about the relaxation of the undriven model to equilibrium have already been analyzed in considerable detail by past researchers [17,20,21,24], starting from the very early work of Desai and Zwanzig [21], it does not seem necessary to reconsider them here.

On the other hand, since equilibrium properties of the undriven model are important for understanding those of Eq. (5) we are interested in, they deserve to be pointed out. The equilibrium distribution has a functional form given by

$$P_e(x)=R^{-1}\exp-\frac{1}{D}\left[\frac{x^4}{4}-\frac{x^2}{2}+\frac{\theta}{2}(x-x_e)^2\right], \tag{7}$$

where R is the normalization factor, while x_e are the solutions of

$$x_e=\int_{-\infty}^{\infty}dx P_e(x)x. \tag{8}$$

It is a simple matter to show that there is a critical line in the parameter space defined by [21]

$$\sqrt{2D_c}=\theta D_{-3/2}(z_c)/D_{-1/2}(z_c), \tag{9}$$

where $z=(\theta-1)/\sqrt{2D}$, and where $D_\nu(z)$ is a parabolic cylinder function, such that below this line, $|z|<|z_c|$ $(D>D_c)$, there exists only one stable equilibrium distribution with $x_e=0$, while above it, $|z|>|z_c|$ $(D<D_c)$, there are two stable equilibrium solutions with $\langle x\rangle_e=\pm x_e$, besides the zero ($\langle x\rangle_e=0$) unstable one. Thus, at the critical line, there is a bifurcation of the equilibrium distribution function. For $D<D_c$ it is always single peaked, while for $D>D_c$, the stable equilibrium distribution has either one or two maxima depending on whether θ is larger than or less than 1.

A few years ago, Shiino proved an H theorem for the above truly nonlinear Fokker-Planck equation in the absence of the driving field [11]. It states that in the long-time limit, the system always reaches one of the equilibrium solutions. Clearly, for a given θ and $D>D_c$, the equilibrium is unique regardless of the initial condition. On the other hand, for $D<D_c$ there are two stable equilibrium solutions and, as t goes to infinity, the system approaches one or the other depending upon the sign of $\langle x(0)\rangle$, or in other words, upon the initial preparation of the system. In this sense, one can say that the functional nonlinearity breaks the ergodicity of the process.

Unfortunately, even the simple system described here becomes extremely complicated if $A\neq 0$, particularly when considering its quasiequilibrium properties. It is the functional nonlinearity of the Fokker-Planck equation which prevents us from making use of the Floquet theory. On the other hand, when a periodic external field is present, we have not been able to extend Shiino's H theorem. In this regard, it is of particular interest to check the validity of both concepts numerically. If they turn out to be wrong one may conclude that there is no simple extension of these concepts to truly nonlinear driven systems. Otherwise, if no anomalous phenomena are observed, we may expect that such an extension does exist at least for this particular problem. In the latter case, it makes sense to study quasiequilibrium solutions of Eq. (5) in more detail with the aim to reveal their properties or relations that would be useful for both understanding the problem and treating it systematically in a simple *analytical* way.

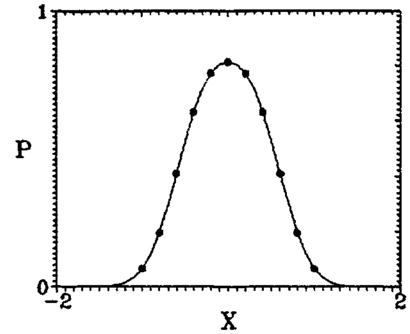

FIG. 1. Time-averaged asymptotic probability [Eq. (13)] for a test process [Eq. (10)] for $\gamma=\theta=A=\Omega=1$, and $D=0.1$. Solid line, exact results obtained from Eq. (12); circles, evaluation using a finite-difference method.

But before presenting our results it is worthwhile to illustrate the power of the method employed in yielding precise quasiequilibrium solutions on a truly nonlinear exactly solvable model driven by a periodic external field. A benchmark model to test numerical methods is a generalized Ornstein-Uhlenbeck process given by

$$\partial_t P(x,t)=\partial_x[\gamma x+\theta\langle x(t)\rangle+A\cos\Omega t+D\partial_x]P(x,t), \tag{10}$$

where γ, θ, A, Ω, and D are constants, while the moment $\langle x(t)\rangle$ is defined by Eq. (6). It is a simple matter to prove that the fundamental solution of Eq. (10), satisfying the initial condition

$$P(x,t|x_0)=\delta(x-x_0), \tag{11}$$

is unique and has the form

$$P(x,t|x_0)=[2\pi\sigma(t)]^{-1/2}\exp\left(-\frac{[x-\langle x(t)\rangle]^2}{2\sigma(t)}\right), \tag{12}$$

$$\langle x(t)\rangle=x_0e^{-(\gamma+\theta)t}+\frac{A[e^{-(\gamma+\theta)t}\sin\varphi-\sin(\Omega t+\varphi)]}{[(\gamma+\theta)^2+\Omega^2]^{1/2}},$$

$$\tan\varphi=\frac{\gamma+\theta}{\Omega},$$

$$\sigma(t)=\frac{D}{\gamma}(1-e^{-2\gamma t}).$$

In Fig. 1 the time-averaged asymptotic probability in x, i.e.,

$$\overline{P_e}(x)=\frac{1}{T}\int_0^T ds P_\infty(x,t+s), \tag{13}$$

obtained from Eq. (12) for $\gamma=\theta=A=\Omega=1$, and $D=0.1$ is shown and compared with that obtained by means of the finite-difference method. Our results were calculated with $K=1$. A grid of 41 points was found to be sufficient to propagate the distribution function in time until quasiequilibrium is reached. It is seen that a very accurate description of the asymptotic regime is already attained in just the lowest order approximation of the present technique, i.e., with $K=1$.

III. FLOQUET THEORY

In order to check the validity of the Floquet theory and the H theorem for the truly nonlinear process governed by Eq. (5), computations were carried out with different initial conditions in a wide range of parameters, namely, $0.1\leqslant\theta\leqslant 2$, $0.05\leqslant D\leqslant 1$, $0\leqslant A\leqslant 0.5$, and $0.01\leqslant\Omega\leqslant 1$. One can mention here that in the absence of the driving field and with $\theta\leqslant 1$, the height of the potential barrier below the critical line is $\frac{1}{4}(\theta-1)^2$, while the largest relaxation time is of order

$$\tau=\frac{\pi}{\sqrt{2}}(1-\theta)^{-1}\exp(-z^2/2). \tag{14}$$

We have found, first, that after some transient period the system always reaches quasiequilibrium, and, second, that depending on θ, D, A, and Ω one or two quasiequilibrium solutions exist, as is the case for the undriven system as well.

In all the cases studied we find that the relation (3) holds within the accuracy of the method used for all quasiequilibrium solutions which may exist. No kind of aperiodicity in the temporal evolution of $P_\infty(x,t)$ and $\langle x(t)\rangle_\infty$ has been observed. Both quantities show oscillations of frequency Ω. It is interesting to note that the same is true for the averages $\langle x^{2k+1}(t)\rangle_\infty$, that are all oscillating with frequency Ω, while the averages $\langle x^{2k}(t)\rangle_\infty$ oscillate either with frequency Ω or 2Ω, depending on θ, D, and A. For small values of A they usually oscillate with Ω, while the frequency is 2Ω otherwise. Fourier analysis of the averages shows a very rapid convergence of the series

$$\langle x^k(t)\rangle_\infty=\int dx P_\infty(x,t)x^k$$
$$=\sum_{n=0} a_n\cos(n\Omega t+\varphi_n), \tag{15}$$

so that two or three terms of this series are usually sufficient to attain a very accurate description. We note, in particular, that a simple approximation of the form

$$\langle x^k(t)\rangle_\infty^F=a_0+a_j\cos(j\Omega t+\varphi_j), \tag{16}$$

where $j=1$ or 2, depending on k and A (see in the above), is found to work well in most cases of interest.

Since the conclusions drawn in all the cases considered are essentially the same, only the results for $\theta=0.5$, $D=0.125$ $(z=1)$, $A=0.2$, and $\Omega=0.1$ are shown in Figs. 2 and 3, where we plot the time evolution of the first two moments, and $P_\infty(x,t)$, respectively. For comparison, we also show in Fig. 2 the approximation

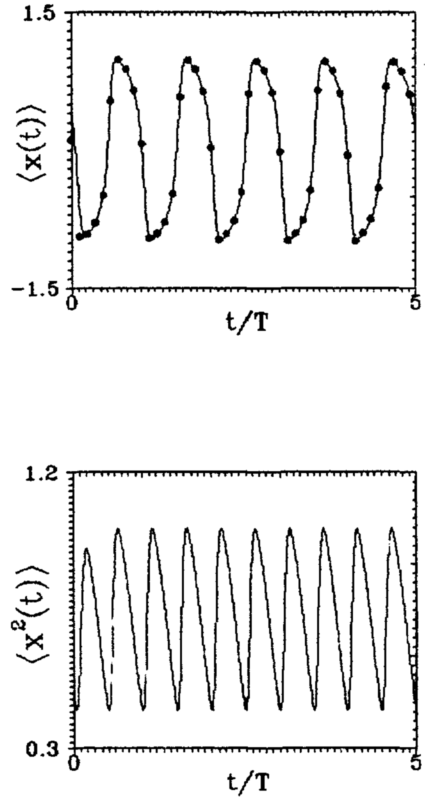

FIG. 2. Temporal evolution of $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$ for the model (5) for $\theta=0.5$, $D=0.125$ $(z=1)$, $A=0.2$, and $\Omega=0.1$. Solid line, numerically exact results; circles, approximation (17).

$$\langle x(t)\rangle_\infty^F=1.0378\cos(\Omega t+1.618)+0.186\cos(3\Omega t+0.991)$$
$$+0.0589\cos(5\Omega t+0.216), \tag{17}$$

which is obtained from the Fourier analysis of the average response. This is just that rare case when three terms of the series in Eq. (15) are necessary to approximate well $\langle x(t)\rangle_\infty$. As evidenced by Figs. 2 and 3, after some transient period $t\sim T$ the temporal evolution becomes obviously periodic with a frequency equal to the driving frequency, while the average response $\langle x(t)\rangle$ shows an oscillation with an amplitude larger than the driving amplitude. It is seen also that Eq. (17) very soon turns out indistinguishable from the numerically exact results. Our findings allows us to conclude that the Floquet theory is valid for the truly nonlinear process (5) at least in its long-time limit form (3). This is the first principal result of the present work.

It is in contrast to the results of [19] where it was found that the system's response is not periodic in time for $\theta=0.5$, $A=0.2$, and $\Omega=0.1$ in the interval $0.69<z<1.2$. A simple reason for this seems to be the fact that the ''phenomenon'' of aperiodicity was met just in the vicinity of the critical line in which some kind of critical slowing down occurs. In the absence of external driving, this is shown to manifest in the form of the divergence of relaxation time on

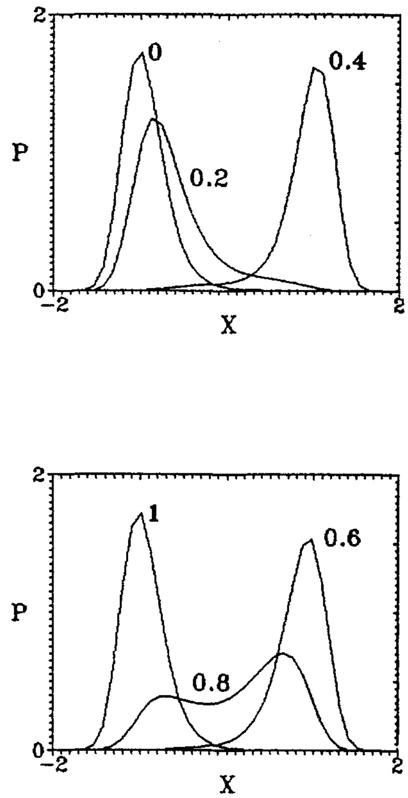

FIG. 3. Same as in Fig. 2, but for the quasiequilibrium distribution function $P_\infty(x,t+rT)$ for $r=0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$, and 1.

approaching phase-transition points [11]. One may expect that the same would be true for the driven system as well. Indeed, with our numerical method, we have found, first, that in the case considered, there is a shift of the critical line with respect to the undriven case from $z_c(A=0)=0.69$ to $z_c(A=0.2)=1.18$. And, second, the relaxation time necessary for the nonlinear driven system to reach quasiequilibrium in the vicinity of the phase-transition points is usually larger by a factor of 2 than the one needed away from the critical line. As a result, stochastic simulations converge very slowly in the neighborhood of the critical line, and one needs to generate too many trajectories to get a good estimate for $P_\infty(x,t)$. For completeness, we also show in Fig. 4 the behavior of the amplification factor, which is the ratio between the amplitude of $\langle x(t)\rangle_\infty$ and that of the external signal A, with respect to z for $\theta=0.5,\ A=0.2$, and $\Omega=0.1$, and compare our results with those obtained by a stochastic simulation technique [19]. As it is seen from Fig. 4, there is a plateau in the interval $0.69<z<1.2$, in which the maximum of the amplification factor is achieved. Our results are in qualitative agreement with those of stochastic simulations. This agreement is not surprising, as the amplification factor turns out to be rather insensitive to the periodicity of the system's response. By this we mean that a more or less accurate estimate of the amplification factor can be obtained even for the first oscillation $t\leqslant T$ when the temporal evolution is not really periodic because quasiequilibrium $P_\infty(x,t)$ is not reached yet. It must also be pointed out that the amplification factor of the truly nonlinear system is larger than that of the usually studied case corresponding to $\theta=0$, Eq. (2).

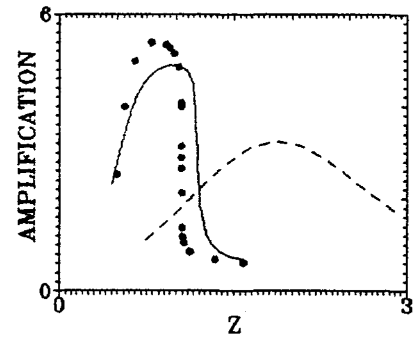

FIG. 4. Amplification factor as a function of z for $\theta=0.5$, $A=0.2$, and $\Omega=0.1$. Solid line, results obtained by a finite-difference method; circles, results of stochastic simulations [19]. Dashed line, results for the linear driven system $\theta=0$.

IV. *H* THEOREM

As we have already emphasized, in all the cases studied we find that with increasing t the system always reaches quasiequilibrium. Moreover, in most cases of interest, the average long-time response is well approximated by the first two terms of the series in Eq. (15), i.e.,

$$\langle x(t)\rangle_\infty=\overline{x_e}+A|\chi|\cos(\Omega t+\varphi), \tag{18}$$

where

$$\overline{x_e}=\int_{-\infty}^{\infty}dx\overline{P_e}(x)x, \tag{19}$$

and where χ is a function of $\theta,\ A,\ \Omega$ and D, which can be interpreted as the generalized susceptibility, while the modulus $|\chi|$ is nothing else than the amplification factor.

We note further that there is a critical surface dividing the parameter space (θ,D,A,Ω) into two regions, so that in one region the system has only one quasiequilibrium solution with $\overline{x_e}=0$, while in the second region there are two possible steady-state values of $\overline{x_e}$. As a result, in the first region the quasiequilibrium is unique regardless of the initial condition. While in the second region there are two quasiequilibrium solutions and, as t goes to infinity, the system approaches one or the other depending upon the value of $\langle x(0)\rangle$. Both situations are clearly illustrated by Fig. 5 which shows the temporal evolution of the system's response $\langle x(t)\rangle$ with different initial conditions. Except for this kind of sensitivity to the initial preparation, inherent to the undriven system as well [20,24], no other nonergodicity phenomena that could be expected due to the large driving amplitude and/or frequency, have been revealed. With these observations one may conclude that there is a straightforward extension of Shiino's H theorem to the truly nonlinear system driven by a periodic external field. This is the second principal result of the present work.

Therefore, it is of particular interest to study in detail the long-time limit properties of the driven system. The necessary computational work has been carried out in a wide range of parameters. The main finding is that these properties are very similar to the equilibrium properties of the undriven system, but there is a shift of the critical surface

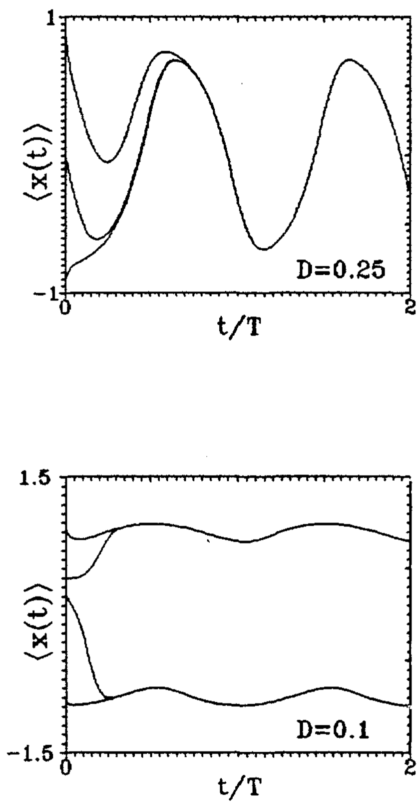

FIG. 5. Temporal evolution of the system's response $\langle x(t)\rangle$ for $\theta=0.5$, $A=0.1$, $\Omega=0.1$ calculated with different initial conditions below ($D=0.25$, $z=0.71$, $\overline{x_e}=0$), and above ($D=0.1$, $z=1.12$, $\overline{x_e}=\pm 0.9$) the critical line.

$$z_c(\theta,A,\Omega)=z_c^0(\theta)+\Delta z(\theta,A,\Omega),$$

where $z_c^0(\theta)$ is defined by Eq. (9). As we have been unable to calculate $\Delta z(\theta,A,\Omega)$ in the whole range of parameters we have calculated this shift for some fixed values of A and Ω as a function of θ. So, by critical line we will mean in the following $z_c(\theta)=z_c^0(\theta)+\Delta z(\theta)$, calculated for fixed A and Ω.

Before presenting our results on the properties of the long-time limit solutions of the driven system, it must be pointed out that these can be analyzed more systematically and conveniently if one proceeds from the quasiequilibrium solutions $P_\infty(x,t)$, satisfying the time-dependent Fokker-Planck equation (5), to their time averages defined by Eq. (13). In the following, we will refer to equilibrium properties of the driven system as meaning those of the time-averaged asymptotic solution $\overline{P_e}$. The most appealing feature of this representation is that the time-averaged asymptotic solutions of Eq. (5) turn out to be time-independent, which is not surprising in view of Eq. (3). The corresponding stationary Fokker-Planck equation reads

$$\partial_x[x^3-x+\theta(x-\overline{x_e})-f(x)+D\partial_x]\overline{P_e}(x)=0, \tag{20}$$

where $f(x)$ is defined by

$$f(x)\overline{P_e}(x)=\frac{1}{T}\int_0^T ds P_\infty(x,t+s)\{\theta[\langle x(t+s)\rangle_\infty-\overline{x_e}] -A\cos\Omega(t+s)\}. \tag{21}$$

The major problem remaining is to determine the function $f(x)$. Unfortunately, we have been unable to evaluate it analytically. We have found numerically that $f(x)$ is antisymmetric with respect to $x=\overline{x_e}$, and well approximated by

$$f(x)=S(x)(x-\overline{x_e}), \tag{22}$$

with the slope S being a function of θ, D, A, and Ω, which varies slowly with x in the vicinity of peaks of the distribution function. Therefore, we introduce a negligible error if one considers S independent of x. This is the third principal result of the present work. It means that time-averaged asymptotic solutions of the driven system are described by the equation

$$\overline{P_e}(x)=R^{-1}\exp-\frac{1}{D}\left[\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}(\theta-\overline{S})(x-\overline{x_e})^2\right], \tag{23}$$

where the shift $\overline{S}$ is chosen independent of x and determined from

$$\overline{S}=\int_{-\infty}^{\infty}dx\overline{P_e}(x)f'(x). \tag{24}$$

One can easily see that the above equation is quite similar to that of the undriven system [cf. Eq. (7)]. It must also be pointed out that the very same result is obtained if one determines $f(x)$ not from Eq. (21), but by approximating the numerically exact $\overline{P_e}(x)$ according to Eq. (23).

The typical form of the time-averaged asymptotic distribution function $\overline{P_e}(x)$, as well as of the dependence S of x, obtained numerically below ($D=0.25$), at ($D=0.175$), and above ($D=0.1$, $\overline{x_e}=\pm 0.9$) the critical line, is shown in Fig. 6 for $\theta=0.5$, $A=0.1$, $\Omega=0.1$. For the sake of comparison we also show in the figure the equilibrium distribution of the shifted undriven system given by Eq. (23). As it is evidenced by Fig. 6, the equilibrium distribution functions are indistinguishable from each other, and S is a rather slowly varying function of x. In the following we will characterize the shift by its average value evaluated in Eq. (24), but we will omit the bar on top.

It is thus seen that the effect of the periodic external field results in the shift S. The latter is a function of the problem parameters which is yet to be determined. This is especially pleasing, since the temporal evolution of the driven and undriven systems are very different from each other. It is worthwhile noticing also that if S were precisely independent of x, Eq. (23) would automatically mean a weak extension of Shiino's H theorem in the sense that this theorem would appear to be valid for the time-averaged asymptotic distribution of the driven system.

With the above results it is not difficult to analyze the equilibrium properties of the driven system. These are wholly determined by S which is generally a function of θ, D, A, and Ω. As we are interested, however, in the ef-

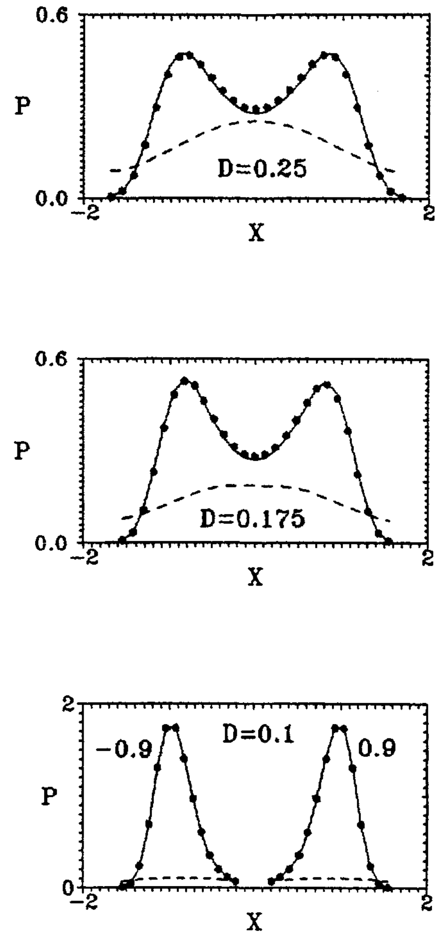

FIG. 6. Time-averaged asymptotic probability $\overline{P_e}$ [Eq. (13)], and shift S [Eqs. (21) and (22)] as functions of x calculated for $\theta=0.5$, $A=0.1$, $\Omega=0.1$ below $(D=0.25)$, at $(D=0.175)$, and above $(D=0.1,\ \overline{x_e}=\pm 0.9)$ the critical line. Circles are the results for the shifted undriven system given by Eqs. (23) and (24).

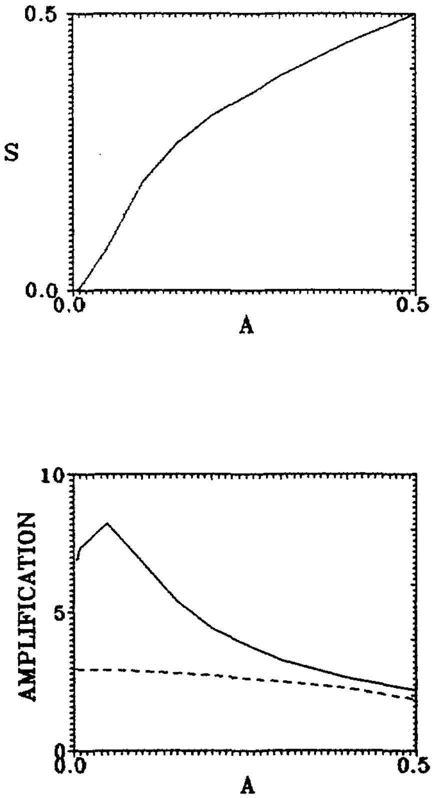

FIG. 7. Shift S, Eq. (24), and amplification factor as a function of the driving amplitude A for $D=0.25$, $\theta=0.5$, and $\Omega=0.1$. Dashed line, results for the linear driven system $\theta=0$.

fect of the external field, we will focus our studies on the behavior of the shift with A and Ω. To begin with, let us consider S as a function of A regarding Ω fixed and finite. There are two distinct limits in the dependence S on A. At $A=0$ the system becomes undriven and nothing interesting happens. At very small but finite A's, the shape of the quasiequilibrium distribution remains very similar to that of the undriven system everywhere except at the stationary points, determined from $U'(x)=0$. In the vicinity of these points $U'(x)$ is comparable with A. This leads to weak oscillations of the maxima of the distribution function, as well as of the averages $\overline{\langle x^k(t)\rangle_\infty}$ with one and the same frequency Ω, and with $\overline{x_e}=x_e$. This is clearly the region of validity of linear response theory, when the probability density can be expanded around the equilibrium solution of the undriven system. With increasing amplitudes of the external field, the regions where $U'(x)$ is comparable with A become broader. Thus, there exist oscillations not only of the maxima of the distribution function themselves, but also of their location, so that $\overline{x_e}$ is no longer equal to x_e. Meanwhile the principal frequency of even powers averages $\langle x^{2k}(t)\rangle_\infty$ appears to be 2Ω. This is the region where linear response theory becomes inadequate. Moreover, in the close vicinity above the critical line, two disconnected equilibrium solutions of the undriven system turn out to be connected due to the external field, so that the system switches between them. The resulting long-time limit solution in this region is unique and corresponds to $\overline{x_e}=0$, while the critical line appears to be shifted. The shift increases with increasing A, and in the limit of infinitely large driving amplitudes the behavior becomes wholly determined by the external field, and no bifurcation of the long-time limit solution occurs. The above observations are illustrated in Fig. 7, which shows the shift S as a function of the driving amplitude A for $D=0.25$, $\theta=0.5$, and $\Omega=0.1$. It is seen that the dependence $S(A)$ is not linear as one might expect from Eqs. (18) and (21). For completeness, the amplification factor is also shown in the figure and compared with that of the corresponding linear system $(\theta=0)$. It is seen that it increases with decreasing A, and yet the amplification factor of the truly nonlinear system is much larger than that with $\theta=0$. Finally, it must be pointed out that the results obtained for the shift S are in agreement with those obtained for the shift of the critical line $\Delta z(\theta)$. The latter are shown in Fig. 8.

The behavior of S with Ω is simpler. For $\Omega=0$ the system becomes undriven. Its equilibrium properties are slightly different from those of Eq. (7) and given by

$$P_e(x)=R^{-1}\exp-\frac{1}{D}\left[\frac{x^4}{4}-\frac{1}{2}x^2+\frac{\theta}{2}\left(x-x_e+\frac{A}{\theta}\right)^2\right]. \tag{25}$$

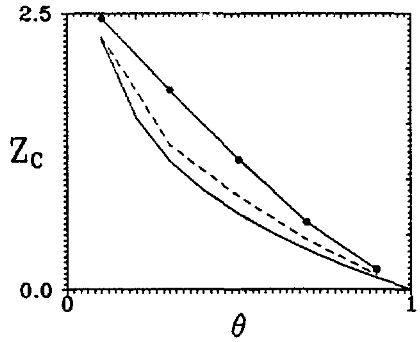

FIG. 8. Equilibrium phase diagram for the model (5) for $\Omega=0.1$. Solid line, results for the critical line of the undriven system $A=0$, Eq. (9). The dashed and circles connected by solid lines are the results for $A=0.1$, and $A=0.2$, respectively.

On the other hand, for each fixed value of the amplitude, in the limit of frequencies larger than that of interwell relaxation, the system becomes insensitive to the external field even for finite amplitudes, because of too fast oscillations, and, therefore, its equilibrium properties become indistinguishable from those of Eq. (7). One may thus expect that in this limit, linear response theory would also be valid whatever the fixed amplitude of the external field considered. In Fig. 9 we show the behavior of the shift S and amplification factor with the driving frequency Ω. It is seen that the shift rapidly decreases from its maximal value at very small

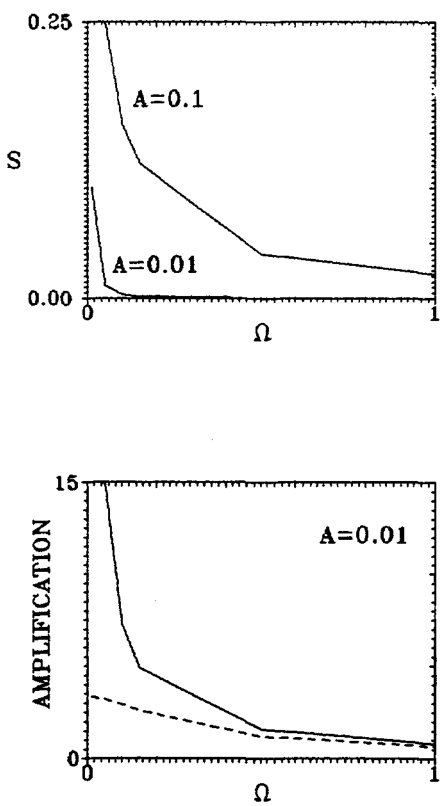

FIG. 9. Shift S, Eq. (24), and amplification factor as a function of the driving frequency Ω for $D=0.25, \theta=0.5$, and $A=0.01$, and 0.1. Dashed line, results for the linear driven system $\theta=0$.

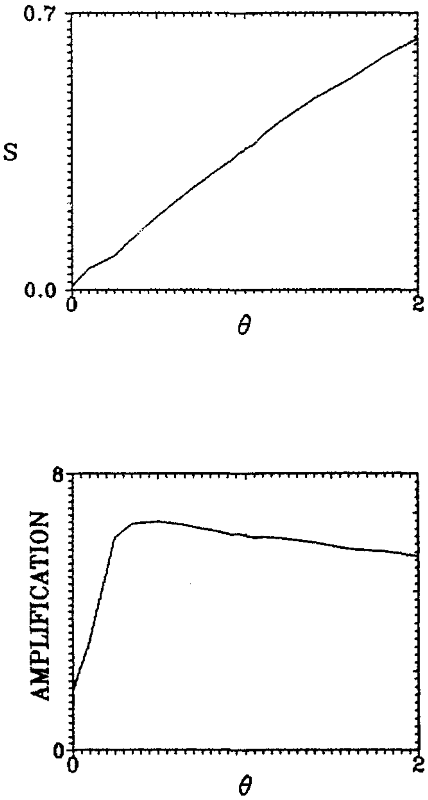

FIG. 10. Shift S, Eq. (24), and amplification factor as a function of θ for $D=0.175$, $\Omega=0.1$, and $A=0.1$.

Ω's, to zero at $\Omega>1$, and yet the amplification factor increases considerably with decreasing the driving frequency. For comparison, we also show in the figure the amplification factor of the corresponding linear system ($\theta=0$). This factor is seen to be much smaller than that for $\theta>0$.

Since the resonance properties of the truly nonlinear system turn out to be very sensitive to the parameter of nonlinearity θ, it makes sense to show the typical behavior of S and amplification factor with θ. As it is seen from Fig. 10, the dependence of the shift on θ is almost linear. The amplification factor reaches its maximal value in the neighborhood of the critical line, and then slowly decreases with increasing θ.

V. LINEAR RESPONSE THEORY

Finally, we check the validity of linear response theory by comparing our results for $|\chi|$ [see Eq. (18)] with those obtained in Ref. [16] through Eq. (4). From the very beginning it is clear that these calculations should be correct, as they were performed with amplitudes of the external field smaller than the height of the potential barrier $(1-\theta)^2/4$. Indeed, as evidenced by Fig. 11, the agreement between the results is excellent. This means that the perturbation method suggested in Ref. [16] offers a very simple tool for describing properties of the truly nonlinear system without solving the corresponding Fokker-Planck equation. We also note that in all the cases considered, the maximum in the dependence of the amplification factor with D and/or z occurs in the close vi-

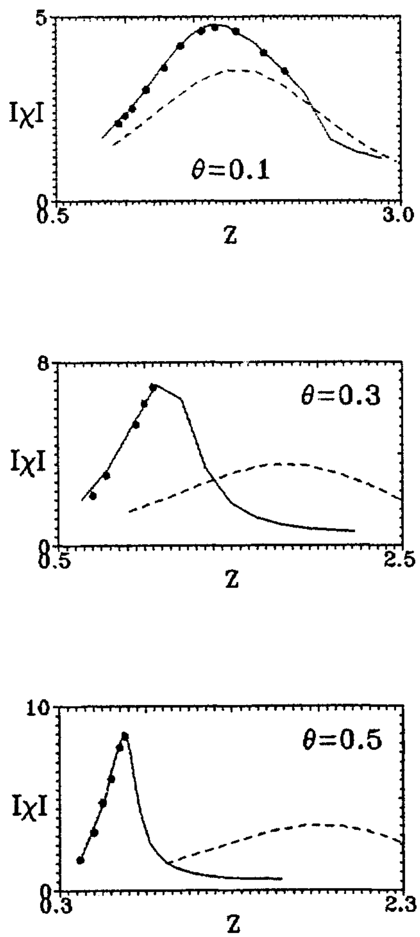

FIG. 11. Modulus of the dynamical susceptibility $|\chi|$, Eq. (18), as a function of z for $\Omega=0.1$ and several values of $\theta=0.1$ ($A=0.032$), $\theta=0.3$ ($A=0.019$), and $\theta=0.5$ ($A=0.01$). Solid line, numerically exact results obtained by means of a finite-difference method; circles, evaluation using Eq. (4); dashed line, results for the linear driven system with $\theta=0$.

cinity above the critical line where the system is already monostable. We again emphasize that the amplification factor of the truly nonlinear system is usually much larger than that of the corresponding linear system ($\theta=0$).

VI. CONCLUSIONS

In this paper, we have studied numerically the combined effects of noise and time periodic external forces in a truly nonlinear system which exhibits a phase transition. The system is described by a Fokker-Planck equation with an explicit dependence of the drift coefficient on the distribution function. To the best of our knowledge, our work is the first attempt to check the validity in this context of the powerful Floquet theory and the H theorem so often applied to linear and/or undriven systems. The principal results obtained are as follows.

(i) No kind of aperiodicity in the temporal evolution of $P_\infty(x,t)$ and $\langle x^k(t)\rangle_\infty$ has been observed. The relation (3) is found to hold within the accuracy of the method used for all quasiequilibrium solutions $P_\infty(x,t)$ that may exist. Our calculations show the validity of the Floquet theory at least in its long-time limit form (3), despite a number of problems ''expected'' from a rigorous point of view. This is an important result indicating that the functional nonlinearity involved in Eq. (5) needs not be an obstacle neither for the use of Eq. (3) in studying stochastic resonance in this system nor for the prediction of its quasiequilibrium properties.

(ii) As t goes to infinity, the system always reaches quasiequilibrium. No kind of nonergodicity, which might be expected due to the large driving amplitude and/or frequency, except for that inherent to the undriven system itself, has been observed. This allows us to conclude that Shiino's H theorem may be extended to the truly nonlinear system driven by a periodic external field.

(iii) What is most remarkable is that the equilibrium properties of the driven system turn out to be quite similar to those of the shifted undriven system, given by Eq. (23). The major unresolved problem is to determine the dependence of the shift on the problem parameters. Determining the shift analytically is particularly efficient when dealing with equilibrium properties of the driven system, such as $\overline{x_e}$, $\overline{P_e(x)}$ and $z_c(\theta,A,\Omega)$. Then, these are no more difficult to evaluate than in the purely undriven case.

(iv) Linear response theory, as expected, is found to be valid for small driving amplitudes. From our analysis one expect that it would also be valid in the limit of large frequencies regardless of the amplitude of the external field.

(v) Of particular importance is the fact that the functional nonlinearity is found to considerably increase the resonance properties of the system under study in the sense that the amplification factor of the truly nonlinear system (5) usually turns out to be much larger than that of the corresponding linear one, Eq. (2), whatever $A>0$.

We believe that the results presented will provide the necessary foundation for further analytical treatment of the problem.

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